REGULARITY FOR A CLASS OF (NON)VARIATIONAL PROBLEMS WITH (NON)STANDARD GROWTH

University of Ottawa Colloquium in Mathematics and Statistics

Sumiya Baasandorj
University of Ottawa
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VARIATIONAL APPROACH: ORLICZ DOUBLE PHASE FUNCTIONAL

3. Orlicz double phase functional (ODP)

$$W^{1,1}(\Omega) \ni v \mapsto \mathcal{P}(v,\Omega) := \int_\Omega [G(|\nabla v|) + a(x)H(|\nabla v|)]\, dx, \tag{1}$$

where

1. $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open subset
2. Sobolev space $W^{1,1}(\Omega) = \{v \in L^1(\Omega) : \text{weak derivative } \nabla v \in L^1(\Omega;\mathbb{R}^n)\}$

$$\int_\Omega v\nabla\varphi\, dx = -\int_\Omega \nabla v \varphi\, dx \quad \forall \varphi \in C_c^\infty(\Omega)$$

3. $\Phi(x,t) := G(t) + a(x)H(t) \quad (x \in \Omega, t \geqslant 0)$
4. $G, H : [0,\infty) \to [0,\infty)$ are N-functions of class $C^1([0,\infty)) \cap C^2((0,\infty))$ such that there exist constants c_G, c_H satisyfing

$$\frac{1}{c_G} \leq \frac{G''(t)t}{G'(t)} \leq c_G \quad \text{and} \quad \frac{1}{c_H} \leq \frac{H''(t)t}{H'(t)} \leq c_H \quad (\forall t > 0) \tag{2}$$

5. $0 \leq a(\cdot) \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1]$.

4. Motivation[1][2]

- Modelling of strongly anisotropic materials
- Elasticity
- Homogenization
- Lavrentiev phenomenon

[1] (V.V. Zhikov, *Lavrentiev phenomenon and homogenization for some variational problems*, C. R . Acad. Sci. Paris. Sér. I Math 316 (1993))

[2] (V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Math. 1986)

5. Musialek-Orlicz space

- We define the complementary function Φ^* of Φ by

$$\Phi^*(x,t) = \sup_{s\geq 0}\left(st - \Phi(x,s)\right)$$

- Musielak-Orlicz space associated to Φ

$$L^{\Phi}(\Omega) := \operatorname{span}\left\{ v : \Omega \to \mathbb{R} : \int_{\Omega} \Phi(x, |v(x)|)\, dx < \infty \right\}.$$

Properties[3]

- If Φ satisfies (2), then $L^{\Phi}(\Omega)$ is a Banach space under the Luxemburg norm

$$\|v\|_{L^{\Phi}(\Omega)} = \inf\left\{ \sigma > 0 : \int_{\Omega} \Phi\left(x, \frac{|v(x)|}{\sigma}\right) dx \leq 1 \right\}$$

.

- If Φ satisfies (2), $L^{\Phi^*}(\Omega)$ is a Banach space under the corresponding Luxemburg norm.

6. Musielak-Orlicz-Sobolev space

- Musielak-Orlicz-Sobolev space associated to Φ

$$W^{1,\Phi}(\Omega) := \operatorname{span}\left\{v \in L^{\Phi}(\Omega) : \text{ weak gradient } \nabla v \in L^{\Phi}(\Omega; \mathbb{R}^n)\right\}.$$

Properties[3]

- If Φ satisfies (2), $W^{1,\Phi}(\Omega)$ is a Banach space under the Luxemburg norm

$$\|v\|_{W^{1,\Phi}(\Omega)} := \|v\|_{L^{\Phi}(\Omega)} + \|\nabla v\|_{L^{\Phi}(\Omega;\mathbb{R}^n)}.$$

- If Φ satisfies (2), $W^{1,\Phi^*}(\Omega)$ is a Banach space.

Example:

- If $\Phi(x,t) = t^p$ $(p > 1)$, then $W^{1,\Phi}(\Omega) = W^{1,p}(\Omega)$ and $\Phi^*(x,t) = t^{\frac{p}{p-1}}$.
- In particular, $\Phi(x,t) = t^2$, then $W^{1,\Phi}(\Omega) = W^{1,2}(\Omega) = H^1(\Omega)$ and $\Phi^*(x,t) = t^2$.

[3](P. Harjulehto and P. Hästö, Orlicz spaces and Generalized Orlicz spaces, Lecture notes in Mathematics, 2019)

7. Principle question

Q1 Discovering suitable optimal conditions to be replaced on nonlinearity under which we prove

Q2 a local minimizer u of ODP functional is regular suitably:

1. Existence of a solution in a given class of functions,
2. Uniqueness of minima satisfying same boundary condition,
3. Boundedness,
4. Harnack's inequality,
5. Hölder continuity,
6. Sobolev regularity,
7. Lipschitz regularity,
8. Gradient Hölder continuity,
9. Smoothness and analiticity if they are obtainable.

Definition of a minimizer[4]

A function $u \in W^{1,1}(\Omega)$ is a local minimizer of the functional $\mathcal{P}$ in (1) if

1. $\Phi(x, |\nabla u|) = G(|\nabla u|) + a(x)H(|\nabla u|) \in L^1(\Omega)$,
2. The minimality condition

$$\int_{\operatorname{supp}(u-w)} \Phi(x, |\nabla u|)\, dx \leq \int_{\operatorname{supp}(u-w)} \Phi(x, |\nabla w|)\, dx$$

is satisfied whenever $w \in W^{1,1}(\Omega)$ is such that $\operatorname{supp}(u - w) \Subset \Omega$.

[4](E. Giusti, Direct methods in the calculus of variations, World Scientific Publishing, 2003)

9. Euler-Lagrange equation

Euler-Lagrange equation

Let $u \in W^{1,1}(\Omega)$ be a local minimizer of ODP functional $\mathcal{P}$. A function $\mathbb{R} \ni t \mapsto f(t) := \mathcal{P}(u + t\varphi, \Omega)$ attains its minimimum at $t = 0$ for every $\varphi \in C_0^\infty(\Omega)$. Then

$$\int_\Omega \langle \frac{G'(|\nabla u|)}{|\nabla u|} \nabla u + a(x) \frac{H'(|\nabla u|)}{|\nabla u|} \nabla u, \nabla \varphi \rangle = 0 \quad (\forall \varphi \in C_0^\infty(\Omega))$$

Definition of a weak solution

$u \in W^{1,1}(\Omega)$ is called a weak solution of Orlicz double phase equation

$$\operatorname{div}\left(\frac{G'(|\nabla u|)}{|\nabla u|} \nabla u + a(x) \frac{H'(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0 \quad \text{in} \quad \Omega \tag{3}$$

if

$$\int_\Omega \langle \frac{G'(|\nabla u|)}{|\nabla u|} \nabla u + a(x) \frac{H'(|\nabla u|)}{|\nabla u|} \nabla u, \nabla \varphi \rangle = 0$$

holds true for all $\varphi \in W_0^{1,1}(\Omega)$ with $\Phi(x, |\nabla \varphi|) \in L^1(\Omega)$.

Theorem 1 (B.-Byun, [Memoirs of AMS 2025])

Let $u \in W^{1,1}(\Omega)$ be a local minimizer of ODP functional $\mathcal{P}$ or a weak solution of ODP equation. If one of the following assumptions

$$\begin{cases} \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + [G(\rho)]^{1+\frac{\alpha}{n}}} < \infty, & \text{(4a)} \\ \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\frac{\alpha\kappa}{n+\kappa}} G(\rho)} < \infty \text{ and } u \in L^{\kappa}(\Omega) & \text{(4b)} \\ \text{or} & \\ \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\alpha} G(\rho)} < \infty \text{ and } u \in L^{\infty}(\Omega) & \text{(4c)} \end{cases}$$

is satisfied, then $u \in C^{0,\theta}_{\mathrm{loc}}(\Omega)$ for some $\theta \in (0,1)$. Moreover, for every $\Omega_0 \Subset \Omega$, we have

$$\|u\|_{L^\infty(\Omega_0)} + [u]_{C^{0,\theta}(\Omega_0)} \le C(\mathbf{data}, \Omega_0)$$

Theorem 2 (B.-Byun, [Memoirs of AMS 2025])

Let $u \in W^{1,1}(\Omega)$ be a nonnegative local minimizer of ODP functional $\mathcal{P}$ or a nonnegative weak solution of ODP equation. If one of the following assumptions

$$\begin{cases} \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + [G(\rho)]^{1+\frac{\alpha}{n}}} < \infty, & \text{(5a)} \\ \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\alpha} G(\rho)} < \infty \text{ and } u \in L^{\infty}(\Omega), & \text{(5b)} \\ \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\frac{\alpha\kappa}{n+\kappa}} G(\rho)} < \infty \text{ and } u \in L^{\kappa}(\Omega), & \text{(5c)} \\ \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\frac{\alpha}{1-\gamma}} G(\rho)} < \infty \text{ and } u \in C^{0,\gamma}(\Omega) & \text{(5d)} \end{cases}$$

is satisfied, for every ball $B_R \subset \Omega_0 \Subset \Omega$, we have

$$\sup_{B_R} u \le c(\mathbf{data}, \Omega_0) \inf_{B_R} u.$$

12. Morrey decay

Theorem 3 (B.-Byun, [Memoirs of AMS 2025])

Let $u \in W^{1,1}(\Omega)$ be a local minimizer of ODP functional $\mathcal{P}$ or a weak solution of ODP equation. If one of the following assumptions

$$\begin{cases} \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + [G(\rho)]^{1+\frac{\alpha}{n}}} < \infty, & \text{(6a)} \\ \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\alpha} G(\rho)} < \infty \text{ and } u \in L^{\infty}(\Omega), & \text{(6b)} \\ \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\frac{\alpha\kappa}{n+\kappa}} G(\rho)} < \infty \text{ and } u \in L^{\kappa}(\Omega), & \text{(6c)} \\ \sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\frac{\alpha}{1-\gamma}} G(\rho)} < \infty \text{ and } u \in C^{0,\gamma}(\Omega) & \text{(6d)} \end{cases}$$

is satisfied, then $u \in C^{0,\theta}_{\mathrm{loc}}(\Omega)$ for every $\theta \in (0,1)$. Moreover, for every $\sigma \in (0,n)$, it holds, whenever $B_r \subset B_R \Subset \Omega$ with $R \le 1$ are concentric balls,

$$\int_{B_r} \Phi(x, |\nabla u|)\, dx \lesssim \left(\frac{r}{R}\right)^{n-\sigma} \int_{B_R} \Phi(x, |\nabla u|)\, dx$$

Theorem 4 (B.-Byun, [Memoirs of AMS 2025])

Let $u \in W^{1,1}(\Omega)$ be a local minimizer of ODP functional $\mathcal{P}$ or a weak solution of ODP equation. If any of the following assumptions

$$
\begin{cases}
\sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + [G(\rho)]^{1+\frac{\alpha}{n}}} < \infty, & \text{(7a)}\\
\sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\alpha} G(\rho)} < \infty \text{ and } u \in L^{\infty}(\Omega), & \text{(7b)}\\
\sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\frac{\alpha\kappa}{n+\kappa}} G(\rho)} < \infty \text{ and } u \in L^{\kappa}(\Omega), & \text{(7c)}\\
\text{or} & \\
\sup\limits_{\rho>0} \dfrac{H(\rho)}{G(\rho) + \rho^{\frac{\alpha}{1-\gamma}} G(\rho)} < \infty \text{ and } u \in C^{0,\gamma}(\Omega) & \text{(7d)}
\end{cases}
$$

is satisfied, then there exists an exponent $\theta \equiv \theta(n, s_G, s_H, \alpha) \in (0,1)$ such that $\nabla u \in C^{0,\theta}_{\mathrm{loc}}(\Omega)$.

SPECIAL FUNCTIONALS

15. Special Double phase functionals

(p, q)-**double phase functional** introduced by Zhikov is the hardest one to treat

$$W^{1,1}(\Omega) \ni v \mapsto \int_{\Omega} \left(|\nabla v|^p + a(x)|\nabla v|^q\right)\, dx, \quad 1 < p \leq q,$$

and its borderline case

$$W^{1,1}(\Omega) \ni v \mapsto \int_{\Omega} \left(|\nabla v|^p + a(x)|\nabla v|^p \log(e + |\nabla v|)\right)\, dx, \quad 1 < p$$

Giuseppe Mingione

FOLLOW

University of Parma
Verified email at unipr.it - Homepage
Calculus of Variations Mathematical analysis Analysis of PDEs Regularity Dark Side

TITLE	CITED BY	YEAR
Regularity results for stationary electro-rheological fluids E Acerbi, G Mingione Archive for Rational Mechanics and Analysis 164, 213-259	812	2002
Regularity results for a class of functionals with non-standard growth E Acerbi, G Mingione Archive for Rational Mechanics and Analysis 156 (2), 121-140	683	2001
Regularity for double phase variational problems M Colombo, G Mingione Archive for Rational Mechanics and Analysis 215 (2), 443-496	631	2015
Bounded minimisers of double phase variational integrals M Colombo, G Mingione Archive for Rational Mechanics and Analysis 218 (1), 219-273	546	2015
Regularity for general functionals with double phase P Baroni, M Colombo, G Mingione Calculus of Variations and Partial Differential Equations 57 (2), 62	544	2018

16. References for special double phase functionals

Maria Colombo

Professor, EPFL Lausanne
Verified email at epfl.ch - Homepage
Mathematical analysis

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TITLE	CITED BY	YEAR
Regularity for double phase variational problems M Colombo, G Mingione Archive for Rational Mechanics and Analysis 215 (2), 443-496	638	2015
Bounded minimisers of double phase variational integrals M Colombo, G Mingione Arch. Ration. Mech. Anal 218 (1), 219-273	548	2015
Regularity for general functionals with double phase P Baroni, M Colombo, G Mingione Calculus of Variations and Partial Differential Equations 57 (2), 62	546	2018
Harnack inequalities for double phase functionals P Baroni, M Colombo, G Mingione Nonlinear Analysis: Theory, Methods & Applications 121, 206-222	400	2015
Non-autonomous functionals, borderline cases and related function classes P Baroni, M Colombo, G Mingione St. Petersburg Mathematical Journal 27 (3), 347-379	325	2016
Calderón–Zygmund estimates and non-uniformly elliptic operators M Colombo, G Mingione Journal of Functional Analysis 270 (4), 1416-1478	256	2016

17. Laplacean: $\Phi(x,t) = t^p$ or $G(t)$

- p-Laplace energy or equation[5]:

$$W^{1,1}(\Omega) \ni v \mapsto \int_\Omega |\nabla v|^p \, dx \longleftrightarrow \operatorname{div}(|\nabla v|^{p-2} \nabla v) = 0 \text{ in } \Omega$$

Motivation: Conformal Geometry, Optimal Matching problem (Ladyzheskaya, Uraltseva, Evans, Giaquinta, Giusti, Lewis, Lindqvist, Manfredi, Uhlenbeck, and many others)

- G-Laplace energy or equation[5]:

$$W^{1,1}(\Omega) \ni v \mapsto \int_\Omega G(|\nabla v|) \, dx \longleftrightarrow \operatorname{div}\left(\frac{G'(|\nabla v|)}{|\nabla v|} \nabla v \right) = 0 \text{ in } \Omega$$

(Diening, Stroffolini, Verde, Lieberman, Byun, and many others)

[5](G. Mingione and V. Radulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity, JMAA, 2021)

18. Many other examples

- $\Phi(x,t) = t^p \log^{\ell}(e+t) + a(x) t^q \log^m(e+t)$ (Zygmund double phase)
- $\Phi(x,t) = G(t) + a(x) G(t) \log(e+t)$
- $\Phi(x,t) = G(t) + a(x) G(t) \log(e+G(t))$ so on...
- Multi-phase functionals such as $\Phi(x,t) = G(t) + a(x) H_a(t) + b(x) H_b(t)$
- However, what happens if G, H do not satisfy the assumption (2)?
  For instance, double phase functional at linear growth given by

$$W^{1,1}(\Omega) \ni v \mapsto \int_{\Omega} \left(|\nabla v| \log(e + |\nabla v|) + a(x) |\nabla v|^q \right) dx$$

  is investigated, see a recent reference [6][7].

[6](G. Mingione and C. De Filippis, Regularity for double phase problems at nearly linear growth, ARMA, 2023)

[7](G. Mingione and C. De Filippis, *Nonuniformly elliptic Schauder theory,* Invent. Math. 2023)

19. Key steps for proving Theorem 1-4

1. Absence of Lavrentiev phenomenon.
2. Sobolev-Poincaré type inequalities.
3. Caccioppoli type inequality.
4. Local boundedness of minima.
5. Almost standard Caccioppoli inequality.
6. Hölder continuity of minima ($C^{0,\theta}_{\mathrm{loc}}$-regularity for some $\theta \in (0,1)$).
7. Harnack's inequality.
8. Harmonic type approximation.
9. Morrey Decay estimates ($C^{0,\theta}_{\mathrm{loc}}$-regularity for any $\theta \in (0,1)$).
10. $C^{1,\theta}_{\mathrm{loc}}$-regularity of minima.

STARTING POINT: DE GIORGI-NASH-MOSER THEORY

21. Hilbert's 19th problem of regularity

Let us consider the following functional

$$v \mapsto \mathcal{F}(v) := \int_\Omega F(\nabla v)\, dx,$$

where $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is analytic, convex and $\det \nabla^2 F > 0$.

- [1] Laplace equation: $\Delta u = 0$ corresponds to $F(\xi) = |\xi|^2$
- [1] Minimal surface equation: $\left(\delta_{ij} - \dfrac{u_i u_j u_{ij}}{1+|\nabla u|^2}\right) u_{ij} = 0$ corresponds to $F(\xi) = \sqrt{1+|\xi|^2}$

Hilbert's 19th question

Are minima of the variational integral $\mathcal{F}$ are analytic? or Whether all such Euler-Lagrange equations

$$\operatorname{div}(\nabla F(\nabla v)) = 0$$

admit only analytic solutions?

[1](E. Giusti, Direct methods in the calculus of variations, World Scientific Publishing, 2003)

22. Early answers

- Bernstein, 1904: C^3 solutions are analytic in two dimensional case
- Hopf, Schauder, Caccioppoli, Morrey, Leray, Liechtenstein (many others)
  Final outcome: $C^{1,\alpha} \Longrightarrow$ **Analytic.**
- Using the difference quotient techniques, we can show a minimizer u of the functional

$$v \mapsto \int_\Omega F(\nabla v)\,dx$$

solves Euler-Lagrange equation

$$\operatorname{div}(\nabla F(\nabla u)) = 0$$

and therefore every component $\nabla_s u$ solves

$$\nabla_i(a_{ij}(x)\nabla_j v) = 0 \text{ with } a_{ij}(x) = F_{\xi_i\xi_j}(\nabla u(x))$$

23. Theorem (De Giorgi-Nash-Moser)

Theorem (De Giorgi 1956, Nash 1957)

Let $v \in W^{1,2}(\Omega)$ be a weak solution of the equation

$$\nabla_i \left(a_{ij}(x) \nabla_j v \right) = 0 \text{ in } \Omega,$$

where

$$\nu |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq L |\xi|^2 \quad (0 < \nu \leq L).$$

Then there exists an exponent $\alpha \equiv \alpha(n, L/\nu) \in (0,1)$ such that

$$\sup_{x \in \Omega_0} |v(x)| + \sup_{x,y \in \Omega_0} \frac{|v(x) - v(y)|}{|x-y|^\alpha} \leq c(n, \nu, L, \Omega_0) \, \|v\|_{L^2(\Omega)} \quad (\Omega_0 \Subset \Omega)$$

Remark

De Giorgi's theorem above concerns the regularity for uniformly elliptic linear equations. However, the linearity does not play a role in his proof ☺.

NONVARIATIONAL APPROACH: THEORY OF VISCOSITY SOLUTIONS

25. A short guide to viscosity solutions

The theory of viscosity solutions applies to certain partial differential equations of the form

$$\mathcal{F}(x, u, \nabla u, \nabla^2 u) = 0 \text{ in } \Omega \subset \mathbb{R}^n, \tag{8}$$

where

- $\mathcal{F} \equiv \mathcal{F}(x, r, \xi, M) : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ satisfies monotonicity condition

  $$\mathcal{F}(x, r, \xi, N) \leqslant \mathcal{F}(x, s, \xi, M) \quad \text{whenever} \quad r \leqslant s \text{ and } M \leqslant N,$$

  where $r, s \in \mathbb{R}$, $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $M, N \in \mathcal{S}(n)$.
- u stands for a real-valued continuous unknown function defined on Ω
- Gradient of u: $\nabla u = \left(\frac{\partial u}{\partial x_i}\right)$
- Hessian of u: $\nabla^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$

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26. Examples

- Laplace equation: $\mathcal{F}(x, r, \xi, M) = -\operatorname{tr}(M)$

$$\Delta u = 0$$

- p-Laplace equation: $\mathcal{F}(x, r, \xi, M) = -|\xi|^{p-2}\operatorname{tr}(M) + (p-2)|\xi|^{p-4}\operatorname{tr}((\xi \otimes \xi)M)$

$$-|\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\operatorname{tr}((\nabla u \otimes \nabla u)\nabla^2 u) = 0$$

- Hamilton-Jacobi-Bellman or Isaac equations for stochastic control and stochastic differential games.
- Monge-Ampére equation

$$\mathcal{F}(x, r, \xi, M) := \begin{cases} -\det(M) + f(x, r, \xi) & \text{if } M \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$\det(\nabla^2 u) = f(x, u, \nabla u) \quad (u \text{ is convex})$$

- Fully nonlinear elliptic equation: $\mathcal{F}(x, r, \xi, M)$ satisfies

$$\lambda \operatorname{tr}(N) \leq F(x, r, \xi, M) - F(x, r, \xi, M + N) \leq \Lambda \operatorname{tr}(N) \quad (\forall N \geq 0).$$

Definition of a viscosity solution[8]

A continuous function u is called **a viscosity supersolution** of (8) if for any $x_0 \in \Omega$, for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local minimum at x_0, there holds

$$\mathcal{F}(x_0, u(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \geqslant 0.$$

A continuous function u is called **a viscosity subsolution** of (8) if for any $x_0 \in \Omega$, for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum at x_0, there holds

$$\mathcal{F}(x_0, u(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \leqslant 0.$$

We say that $u \in C(\Omega)$ is a viscosity solution of (8) if u is **a viscosity supersolution and subsolution simultaneously.**

[8](M.G. Crandal, H. Ishii and P.L. Lions, Bulletin of the AMS, 1992)

28. A class of nonlinear elliptic equations

We consider a class of singular/degenerate nonlinear elliptic equations

$$\mathcal{F}(x,u,\nabla u,\nabla^2 u) := A(x,,u,\nabla u)F(\nabla^2 u) - f(x) = 0 \tag{9}$$

where

- $A(x,u,\nabla u)|\nabla u|^2 \approx \Phi(x,|\nabla u|) = G(|\nabla u|) + a(x)H(|\nabla u|)$ (Orlicz double phase integrand function)
- $F : \mathcal{S}(n) \to \mathbb{R}$ is an operator satisfying uniform ellipticity condition

$$\lambda \operatorname{tr}(N) \le F(M) - F(M+N) \le \Lambda \operatorname{tr}(N) \quad (\forall 0 \le N \in \mathcal{S}(n))$$

- $f \in L^\infty(\Omega)$

Motivation:

- Transmission problems for diffusion processes in heterogeneous media with applications to
    - thermal
    - electromagnetic conductivity
    - composite materials

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29. Some recent results

Theorem (B.-Byun-K.-A. Lee-S.-C. Lee)

Let $u \in C(\Omega)$ be a viscosity solution of (9) under suitably optimal assumptions on nonlineaty.

- $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$[10].
- The existence of solution and global $C^{1,\alpha}$ estimate with Dirichlet boundary condition[11].

Basic ideas of the proof

- Ishii-Lions techniques and compactness arguments.
- Peron's method for the existence of a solution.

,

[10] B., S.S.Byun, K.A. Lee and S.C. Lee, *$C^{1,\alpha}$-regularity for a class of degenerate/singular fully non-linear elliptic equations*, IFB 2024)

[11] B., S.S.Byun, K.A. Lee and S.C. Lee, *Global regularity results for a class of singular/degenerate fully nonlinear elliptic equations*, MZ 2024

STARTING POINT: KRYLOV-SAFANOV THEOREM

31. Krylov-Safanov theorem and Caffarelli's theorem

Theorem(Caffarelli, [Anal. Math. 1989])

Any $u \in C(\Omega)$ viscosity solution of

$$\mathcal{F}(x, u, \nabla u, \nabla^2 u) := F(\nabla^2 u) = 0 \text{ in } \Omega$$

is in $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for $\alpha \equiv \alpha(n, \nu, L) \in (0, 1]$.

Theorem (Krylov-Safanov [12])

Let $u \in C(\Omega)$ be a viscosity solution to

$$\mathcal{F}(x, \nabla^2 u) := -a_{ij}(x)\partial_{ij} u = 0 \text{ in } \Omega \subset \mathbb{R}^n.$$

where there exist constant $0 < \lambda \leq \Lambda$ such that

$$\lambda|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2 \quad (\forall x \in \Omega,\, \xi \in \mathbb{R}^n)$$

Then we have $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ and it satisfies Harnack's inequality.

[12](N.V. Krylov, Lectures on elliptic and parabolic PDEs in Hölder spaces, Graduate studies in Mathematics AMS, 1996)

MONGE-AMPÉRE EQUATION AND OPTIMAL TRANSPORT

33. Regularity for Monge-Amperé equation

$$\mathcal{F}(x, u, \nabla u, \nabla^2 u) := -\det(\nabla^2 u) + f(x) = 0 \quad \text{in} \quad \Omega$$

where

- u is convex
- $0 < \lambda < f \in L^\infty(\Omega) < \Lambda$.

Theorem (Caffarelli [Ann. Math. 1990])

Let u be a strictly convex function satisfying Monge-Ampére equation above in viscosity sense. Then there exists $\alpha \equiv \alpha(n, \lambda, \Lambda) \in (0,1)$ such that $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$. More precisely, for every $\Omega_0 \Subset \Omega$, there exists a constant C depending only n, λ, Λ, Ω_0 and the modulus of convexity of u such that

$$\sup_{x \neq y \in \Omega_0} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^\alpha} \leq C$$

Remark

There are many other discovered regularity results such as C^∞, $C^{2,\alpha}$, $W^{2,p}$, $W^{2,1}$ so on (Caffarelli, Figalli, De Philippis, Wang, many others)

$$W_2^2(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{\Omega_X\times\Omega_Y} \frac{|x-y|^2}{2}\, d\pi(x,y),$$

where

- the set of transport plans

$$\Pi(\mu,\nu) := \{\pi \in \mathcal{P}(\Omega_X\times\Omega_Y) : \pi(A\times\Omega_Y) = \mu(A) \text{ and } \pi(\Omega_X\times B) = \nu(B)\}$$
$$(\text{ for any } A\subset\Omega_X \text{ and } B\subset\Omega_Y \text{ measurable subsets }).$$

- Open, bounded, and convex sets $\Omega_X, \Omega_Y \subset \mathbb{R}^n$
- Probability measures $\mu \in \mathcal{P}(\Omega_X)$ and $\nu \in \mathcal{P}(\Omega_Y)$
- $\mu = f\mathcal{L}^n$ and $\nu = g\mathcal{L}^n$ with spt $f = \overline{\Omega}_X$ and spt $g = \overline{\Omega}_Y$
- There are positive constants λ and Λ such that

$$\lambda \le f, g \le \Lambda$$

.

Theorem(Brenier [13])

1. $\pi_0 \in \Pi(\mu,\nu)$ is an optimal transport plan for $W_2(\mu,\nu)$ if and only if

$$\begin{aligned}\mathrm{spt}(\pi_0) \subset \mathrm{Graph}(\partial u_0) &:= \{(x,y) \in \Omega_X \times \Omega_Y : y \in \partial u_0(x)\} \\ &= \{(x,y) \in \Omega_X \times \Omega_Y : u_0(x) + v_0(y) = \langle x, y\rangle\}\end{aligned}$$

   for a convex semicontinuous function $u_0 : \Omega_X \to \mathbb{R}$, where
   - $v_0 := u_0^*$ is the Legendre transform of u_0.

2. u_0 is differentiable μ-a.e. and there is a unique optimal transport plan

$$\pi_0 := (\mathrm{Id}, \nabla u_0)_{\#}\mu$$

   for $W_2(\mu,\nu)$, where ∇u_0 is called Brenier's optimal transport map from μ to ν.

[13] Y. Brenier, Polar factorization and monotone rearrangement of vector-valued function, Comm. Pure Appl. Math. 1991

Regularity of Brenier's map

- Convex function $u_0 : \Omega_X \to \mathbb{R} \cup \{+\infty\}$ (defined above) solves the so-called **Monge-Ampére equation**

$$g(\nabla u) \det \nabla^2 u = f$$

  in the sense of Brenier (viscosity sense).
- $u_0 \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ by Caffarelli's regularity theorem for Monge-Ampére equation.

37. Future Research Plan

- Recently, we (with A. Gerolin and S. Di Marino) have obtained $C^{1,\alpha}_{\text{loc}}$ regularity results for potentials of entropy regularized optimal transport problems with quadratic cost. We try to obtain other regularity results for the same potential.
- Seek for potential ways to show regularity of Kantorovich potentials (without Caffarelli's results)
- Barycenter problems, optimal transport with Coulomb cost, and many others. [3]
- Regularity for variational functionals with nonstandard growth, and nonvariational problems.[1]
- Parabolic double phase type problems are being investigated intensively, like [2]

$$u_t - \operatorname{div}\left(\frac{G'(|\nabla u|)}{|\nabla u|}\nabla u + a(x,t)\frac{H'(|\nabla u|)}{|\nabla u|}\nabla u\right) = f$$

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[1](C. De Filippis and G. Mingione, *Nonuniformly elliptic Schauder theory*, Invent. Math., (2023))

[2]J. Kinnunen, K. Moring and K, Wontae, *Gradient higher integrability for degenerate parabolic double-phase systems*, ARMA, 2023

[3]G. Friesecke, *Optimal transport: A comprehensive introduction to Modeling, Analysis, Simulation, Applicaions*, SIAM, 2025

Thank you for your attention ☺.